

# $L_p$ -MULTIPLIERS ON QUANTUM TORI

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ABSTRACT. It was shown by Chen, Xu and Yin that completely bounded Fourier multipliers on noncommutative  $L_p$ -spaces of quantum tori  $\mathbb{T}_\theta^d$  do not depend on the parameter  $\theta$ . We establish that the situation is somehow different for bounded multipliers. The arguments are based on transference from the commutative torus.

## 1. INTRODUCTION

Quantum tori are very natural examples in noncommutative geometry for a long time. Nevertheless not much was done about them from a more analytical point of view. Elements in quantum tori have a formal Fourier expansion. Thanks to it most of the classical objects from harmonic analysis make sense and one is tempted to look for counterparts of basic results. This approach really started with a recent paper [2] by Chen, Xu and Yin. Motivated by the developments of noncommutative harmonic analysis and integration, they established for instance the pointwise convergence of Fejér or Bochner-Riesz means in  $L_p$ . They also considered natural Fourier multipliers and showed that their complete bounded norm on  $L_p$  do not depend on the parameters of the tori. In another article [8], Xiong, Xu and Yin continued this line of investigation by looking at analogues of classical function spaces. Fourier multipliers and transference techniques played an important rôle there. The main goal of this note is to show that the bounded norm of  $L_p$ -multipliers on noncommutative tori somehow depends on the parameters solving a question from [2].

We refer to [7, 5] for results on noncommutative  $L_p$ -spaces associated to (semi-finite) von Neumann algebras and to [6] for operator spaces. We recall the basics on quantum tori, we refer to [3, 2] for more details. Let  $d \geq 1$  and  $\theta$  be a real skew symmetric matrix of size  $d$ . The  $d$ -dimensional noncommutative torus associated to  $\theta$  is the universal  $C^*$ -algebra  $\mathcal{A}_\theta^d$  generated by  $d$  unitary operators  $U_k^\theta$  such that

$$U_k^\theta U_j^\theta = e^{2\pi i \theta_{k,j}} U_j^\theta U_k^\theta.$$

We may drop the exponent  $\theta$  when no confusion can occur. Also when  $d = 2$ , we prefer to think of  $\theta$  as a real number and we use  $U_\theta$  and  $V_\theta$  for the generators. So that we have  $U_\theta V_\theta = e^{2i\pi\theta} V_\theta U_\theta$ .

For a multi-index  $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$ , we put  $U^m = U_1^{m_1} \dots U_d^{m_d}$  for basis monomials. As usual a polynomial is a finite sum of monomials. The set of polynomials is dense in  $\mathcal{A}_\theta^d$ . There is a natural normalized trace  $\tau$  on  $\mathcal{A}_\theta^d$  defined by  $\tau(U^m) = 0$  unless  $m = 0$ . The GNS representation of  $\tau$  gives rise to a finite von Neumann algebra  $L_\infty(\mathbb{T}_\theta^d)$ . We denote by  $L_p(\mathbb{T}_\theta^d)$  the associated  $L_p$ -spaces. Any element in  $x \in L_1(\mathbb{T}_\theta^d)$  has a natural formal Fourier expansion  $x \sim \sum \hat{x}(m) U^m$  where  $\hat{x}(m) = \tau(x(U^m)^*)$ . Given a function  $\varphi : \mathbb{Z}^d \rightarrow \mathbb{C}$ , we say that it is the symbol of a multiplier (resp. completely bounded) on  $L_p(\mathbb{T}_\theta^d)$  if the map defined on monomials by  $M_\varphi(U^m) = \varphi(m) U^m$  extends to bounded (resp. completely bounded) map on  $L_p(\mathbb{T}_\theta^d)$ . We refer to [2, 8] for basic facts on those multipliers. We will use quite often multipliers of Fejér type  $F_n^d : \mathbb{Z}^d \rightarrow \mathbb{C}$  given by  $F_n^d(m_1, \dots, m_d) = \prod_{i=1}^d \left(1 - \frac{|m_i|}{n}\right)^+$ . They give rise to unital completely positive multipliers and  $M_{F_n^d}$  is a completely contractive pointwise approximation of the identity in  $L_p(\mathbb{T}_\theta^d)$ ,  $1 \leq p < \infty$  and in  $\mathcal{A}_\theta^d$  (see Proposition 2.1).

When  $\theta = 0$ , we recover classical function spaces  $L_p(\mathbb{T}_0^d) = L_p(\mathbb{T}^d, dm)$  where  $dm$  is the Haar measure on  $\mathbb{T}^d$  and  $\mathcal{A}_0^d = C^*(\mathbb{Z})$ . In [5] Proposition 8.1.3, Pisier gave the first example of a bounded multiplier on  $L_p(G)$  ( $1 < p \neq 2 < \infty$  and  $G$  any infinite compact abelian group) that is not completely bounded. We aim to use it for Fourier multipliers on  $L_p(\mathbb{T}_\theta^d)$ . The idea behind it is pretty simple : when  $\theta \notin \mathbb{Q}$ ,  $\mathcal{A}_\theta^d$  contains asymptotically a nice copy of  $C^*(\mathbb{Z}) \otimes \mathcal{A}_\theta^d$ .

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The paper is organized as follows. First we give a short proof of the independences on  $\theta$  of completely bounded multipliers in section 2. Next, we look at bounded  $L_p$ -multipliers ( $1 < p \neq 2 < \infty$ ), we explain the basic commutative estimates we need, the construction of ultraproducts and derive the result. We finish by looking at bounded  $L_\infty$ -multipliers.

## 2. COMPLETELY BOUNDED MULTIPLIERS

In this section, we give a short direct proof of the following result from [2]:

**Proposition 2.1.** *Let  $1 \leq p \leq \infty$ , for any  $\varphi : \mathbb{Z}^d \rightarrow \mathbb{C}$ , we have*

$$\|M_\varphi\|_{cb(L_p(\mathbb{T}_\theta^d))} = \|M_\varphi\|_{cb(L_p(\mathbb{T}^d))}.$$

*Proof.* By the universal property of  $\mathcal{A}_{\theta+\theta'}^d$ , one can define a  $*$ -homomorphism  $\pi : \mathcal{A}_{\theta+\theta'}^d \rightarrow \mathcal{A}_\theta^d \otimes \mathcal{A}_{\theta'}^d$  by  $\pi(U_i^{\theta+\theta'}) = U_i^\theta \otimes U_i^{\theta'}$ . It is obviously trace preserving and hence extends to a complete isometry on all  $L_p$ ,  $1 \leq p \leq \infty$ . Since  $(Id \otimes M_\varphi) \circ \pi = \pi \circ M_\varphi$ , it follows that  $\|M_\varphi\|_{cb(L_p(\mathbb{T}_{\theta+\theta'}^d))} \leq \|M_\varphi\|_{cb(L_p(\mathbb{T}_\theta^d))}$  for any  $\theta$  and  $\theta'$  from which equality follows.  $\square$

For any  $\varphi : \mathbb{Z}^d \rightarrow \mathbb{C}$ , let  $s_\varphi$  be the (possibly unbounded) linear functional on  $\mathcal{A}_\theta^d$  be given on monomials by  $s_\varphi(U^m) = \varphi(m)$ ,  $m \in \mathbb{Z}^d$ .

The preceding proposition can be extended to

**Proposition 2.2.** *Let  $1 \leq p \leq \infty$ , for any  $\varphi : \mathbb{Z}^d \rightarrow \mathbb{C}$ , we have*

$$\|M_\varphi : L_p(\mathbb{T}_\theta^d) \rightarrow L_p(\mathbb{T}_\gamma^d)\|_{cb} = \|M_\varphi : L_p(\mathbb{T}_{\theta-\gamma}^d) \rightarrow L_p(\mathbb{T}^d)\|_{cb}.$$

Moreover

$$\|M_\varphi : L_\infty(\mathbb{T}_\theta^d) \rightarrow L_\infty(\mathbb{T}_\gamma^d)\|_{cb} = \|s_\varphi\|_{(\mathcal{A}_{\theta-\gamma}^d)^*}.$$

*Proof.* The proof of the first part is the same as above. For the second one, assume  $M_\varphi : L_\infty(\mathbb{T}_{\theta-\gamma}^d) \rightarrow L_\infty(\mathbb{T}^d)$ . Composing it with a Fejér type kernel  $F_n^d$ ,  $M_{F_n^d \varphi} : L_\infty(\mathbb{T}_{\theta-\gamma}^d) \rightarrow C^*(\mathbb{Z}^d)$ . Composing again with the trivial character, we get that  $\|s_{F_n^d \varphi}\|_{(\mathcal{A}_{\theta-\gamma}^d)^*} \leq \|M_\varphi\|$ . Letting  $n \rightarrow \infty$  gives the first part. For the reverse inequality, if  $\|s_\varphi\| < \infty$  then by composition with  $M_{F_n^d}$ ,  $\|s_{F_n^d \varphi}\| \leq \|s_\varphi\|$ , using the  $*$ -homomorphism  $\pi : \mathcal{A}_\theta^d \rightarrow \mathcal{A}_\gamma^d \otimes \mathcal{A}_{\theta-\gamma}^d$  and composing it with  $Id \otimes s_{F_n^d \varphi}$ , we get that  $\|M_{F_n^d \varphi} : L_\infty(\mathbb{T}_\theta^d) \rightarrow L_\infty(\mathbb{T}_\gamma^d)\| \leq \|s_\varphi\|$ . But  $M_{F_n^d \varphi}$  is then a family of normal uniformly bounded maps converging pointwise for the  $w^*$ -topology. Its limit is  $M_\varphi$ .  $\square$

## 3. BOUNDED MULTIPLIERS ON $L_p$ , $1 < p < \infty$

**3.1. Periodization of Fourier multipliers.** We aim to prove a periodization result for Fourier multipliers on  $\mathbb{T}$ . We proceed by using some folklore discretization arguments.

Denote by  $z$  the canonical generator of  $L(\mathbb{Z}) = L_\infty(\mathbb{T})$  and for any  $n \geq 1$ ,  $\gamma_n$  that of  $L(\mathbb{Z}/n\mathbb{Z})$  the group von Neumann algebra associated to the finite group  $\mathbb{Z}/n\mathbb{Z}$ . We always choose the representative of a class in  $\mathbb{Z}/n\mathbb{Z}$  with smallest absolute value.

**Lemma 3.1.** *Let  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a non decreasing convex function with  $f(0) = 1$ , then for any  $n \geq 0$ , there exists a measure  $\mu$  on  $\mathbb{T}$  such that*

$$\forall |k| \leq n, \hat{\mu}(k) = f(|k|) \quad \text{and} \quad \|\mu\| \leq f(n)^2.$$

*Proof.* The sequence given by  $m_k = f(n - k)$  for  $k \leq n$  and  $m_k = f(0) = 1$  for  $k > n$  is non increasing convex, hence there exists a positive measure  $\nu$  with  $\hat{\nu}(k) = m_{|k|}$  with mass  $f(n)$ , see Theorem 3.7 in [1]. Then  $\mu = (z^n \nu) * (z^{-n} \nu)$  has the desired properties.  $\square$

We fix some  $1 \leq p \leq \infty$ . We also denote by  $\mathcal{P}_d = \text{span}\{z^k; |k| \leq d\} \subset L_p(L(\mathbb{Z}))$  and similarly  $\mathcal{P}_d^n = \text{span}\{(\gamma_n)^k; |k| \leq d\} \subset L_p(L(\mathbb{Z}/n\mathbb{Z}))$ .

For  $2d < n$ , one can consider the formal identity  $j_{d,n} : \mathcal{P}_d \rightarrow \mathcal{P}_{d,n}$  given by  $j_{d,n}(z^k) = \gamma_n^k$ .

As usual let  $\sin_c(x)$  be  $\frac{\sin x}{x}$ .

**Proposition 3.2.** *For any  $n > 2d$ ,*

$$\|j_{d,n}\|_{cb} \leq \sin_c^{-2}\left(\frac{d\pi}{n}\right), \quad \|j_{d,n}^{-1}\|_{cb} \leq \left(\left(1 - \frac{2d}{n}\right) \sin_c\left(\frac{d\pi}{n}\right)\right)^{-2}.$$

*Proof.* One can see  $L_p(L(\mathbb{Z}/n\mathbb{Z}))$  as a subspace of  $L_p(L(\mathbb{Z})) = L_p(\mathbb{T})$  by looking at piecewise constant functions. More precisely, put for  $z \in \mathbb{T}$ ,  $g(z) = \omega^l$  where  $\omega^l$  is the closest  $n$ -root of 1 to  $z$ , then  $g$  and  $\gamma_n$  have the same distribution. We identify them.

Let  $\mathcal{E}$  be the conditional expectation onto the algebra generated by  $g \approx \gamma_n$ . Easy computations give that for  $k \in \mathbb{Z}$ ,  $\mathcal{E}z^k = \text{sinc}\left(\frac{k\pi}{n}\right)(\gamma_n)^k$ . The function  $\text{sinc}^{-1}$  is convex on  $[0, \pi[$ , using Lemma 3.1, there exists a measure on  $\mathbb{T}$  with  $\mu(k) = \text{sinc}^{-1}\left(\frac{|k|\pi}{n}\right)$  for  $|k| \leq d$  with mass at most  $\text{sinc}^{-2}\left(\frac{d\pi}{n}\right)$ . For any  $f \in \mathcal{P}_d$ , we have  $j_{n,d}(f) = \mathcal{E}(f * \mu)$ , this gives the first estimate.

Computing the Fourier expansion in  $L_2(\mathbb{T})$  gives for  $|k| < n$

$$(\gamma_n)^k = \sum_{j \in \mathbb{Z}} \text{sinc}\left(\frac{\pi(k+jn)}{n}\right) z^{k+jn}.$$

Hence with the Fejér kernel  $F_{n/2}$ , assuming  $|k| \leq d < n/2$ :

$$F_{n/2} * \gamma_n^k = \left(1 - \frac{2|k|}{n}\right) \text{sinc}\left(\frac{\pi k}{n}\right) z^k.$$

To get  $j_{n,d}^{-1}$  one just need to make corrections as above using that  $\text{sinc}^{-1}$  and  $(1-x)^{-1}$  are convex.  $\square$

**Proposition 3.3.** *Let  $\varphi : \mathbb{Z} \rightarrow \mathbb{C}$  be a Fourier multiplier on  $L_p(\mathbb{T})$ , then there exists a sequence of periodic multipliers  $\varphi_n : \mathbb{Z} \rightarrow \mathbb{C}$  converging pointwise to  $\varphi$  with  $\|M_{\varphi_n}\| \leq \|M_\varphi\|$  and  $\|M_{\varphi_n}\|_{cb} \leq \|M_\varphi\|_{cb}$ .*

*Proof.* Fix  $n > 2d$ . We use transference once again. Consider the measure preserving  $*$ -homomorphism  $\pi_n : L(\mathbb{Z}) \rightarrow L(\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z})$  given by  $\pi_n(z) = z \otimes \gamma_n$ . Let  $\psi : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{C}$  and  $q : \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$  be the quotient mapping. Using the intertwining identity  $(Id \otimes M_\psi) \circ \pi_n = \pi_n \circ M_{\psi \circ q}$  and the fact that bounded maps on  $L_p$  tensorize with the identity of  $L_p$  by Fubini's theorem, one gets that

$$\|M_{\psi \circ q} : L_p(\mathbb{T}) \rightarrow L_p(\mathbb{T})\| \leq \|M_\psi : L_p(\mathbb{T}/n\mathbb{Z}) \rightarrow L_p(\mathbb{T}/n\mathbb{Z})\|.$$

Note that, for  $d < n$ , the Fejér Kernel  $F_d$  is also positive definite on  $\mathbb{Z}/n\mathbb{Z}$ . The map  $j_{n,d} \circ M_\varphi \circ j_{n,d}^{-1} \circ M_{F_d}$  is the Fourier multiplier  $\tilde{\varphi}_{n,d}$  on  $\mathbb{Z}/n\mathbb{Z}$  so that  $M_{\tilde{\varphi}_{n,d}}(\gamma_n^k) = \varphi(k) \left(1 - \frac{|k|}{d}\right) \gamma_n^k$ . Using Proposition 3.2, we get  $\|M_{\tilde{\varphi}_{n,d}}\| \leq c_{n,d} \|M_\varphi\|$  with  $c_{n,d} \rightarrow 1$  as  $n/d \rightarrow \infty$  and similarly with cb norms.

Let  $\varphi_n$  be  $\tilde{\varphi}_{n^2,n}/c_{n^2,n} \circ q$ . Obviously  $\|M_{\varphi_n}\| \leq \|M_\varphi\|$ ,  $\|M_{\varphi_n}\|_{cb} \leq \|M_\varphi\|_{cb}$  and  $\lim_{n \rightarrow \infty} \varphi_n(k) = \varphi(k)$ .  $\square$

**3.2. Ultraproducts.** Given a family of finite von Neumann algebras  $(\mathcal{M}_i, \tau_i)_{i \in \mathbb{N}}$  with normalized trace  $\tau_i$ , their (reduced) von Neumann ultraproduct along the free ultrafilter  $\mathfrak{U}$  is

$$\mathcal{M}_{\mathfrak{U}} = \ell_\infty(\mathcal{M}_i)/I_{\mathfrak{U}}, \quad I_{\mathfrak{U}} = \{(x_i) \in \ell_\infty(\mathcal{M}_i) \mid \lim_{\mathfrak{U}} \tau_i(x_i^* x_i) = 0\}.$$

It is a finite von Neumann algebra with normalized trace  $\tau((x_i)^\bullet) = \lim_{\mathfrak{U}} \tau_i(x_i)$ , see [6] section 9.10.

**Lemma 3.4.** *Let  $T_i : \mathcal{M}_i \rightarrow \mathcal{M}_i$  be a sequence of maps such that  $\sup_i \|T_i\|_{B(\mathcal{M}_i)} + \|T_i\|_{B(L_2(\mathcal{M}_i))} < \infty$ , then  $T_{\mathfrak{U}} = \prod_{\mathfrak{U}} T_i$  is well defined on  $\mathcal{M}_{\mathfrak{U}}$  and moreover  $\|T_{\mathfrak{U}}\|_{B(L_p(\mathcal{M}_{\mathfrak{U}}))} \leq \lim_{\mathfrak{U}} \|T_i\|_{B(L_p(\mathcal{M}_i))}$  (the same holds for cb norms).*

*Proof.* The assumptions yield that  $T_{\mathfrak{U}}$  is well defined on  $\ell_\infty(\mathcal{M}_i)$  and stabilizes  $I_{\mathfrak{U}}$ , so it is well defined on  $\mathcal{M}_{\mathfrak{U}}$ .

For  $x = (x_i)^\bullet \in \mathcal{M}_{\mathfrak{U}}$ , we have  $\|x\|_{L_p(\mathcal{M}_{\mathfrak{U}}, \tau)} = \lim_i \|x_i\|_{L_p(\mathcal{M}_i, \tau_i)}$  by the definition of the trace. The second assertion follows.  $\square$

We now focus on 2-dimensional tori  $\mathbb{T}_\theta^2$ . Recall that the two generators satisfy  $U_\theta V_\theta = e^{i2\pi\theta} V_\theta U_\theta$ . We still denote by  $z$  the generator of  $L_\infty(\mathbb{T})$ .

**Proposition 3.5.** *Fix some integer  $N > 0$ . For  $\theta \notin \mathbb{Q}$  and  $\gamma \in \mathbb{R}$ , there is an explicit cb-isometric embedding*

$$\pi : L_p(\mathbb{T}) \otimes_p L_p(\mathbb{T}_\gamma^2) \rightarrow L_p(\mathbb{T}_\theta^2)_{\mathfrak{U}},$$

*obtained by a  $*$ -representation of the form*

$$\pi(z) = (U_\theta)^\bullet, \quad \pi(U_\gamma) = (U_\theta^{Nl_n})^\bullet, \quad \pi(V_\gamma) = (V_\theta^{k_n})^\bullet.$$

*Proof.* Let  $n > 0$ , since  $\theta \notin \mathbb{Q}$ , one can find some  $k_n$  such that  $|e^{i2\pi k_n \theta} - 1| < \frac{1}{n}$ . Similarly as  $k_n N \theta \notin \mathbb{Q}$ , one can find  $l_n$  such that  $|e^{i2\pi k_n N l_n \theta} - e^{i2\pi \gamma}| < \frac{1}{n}$ . The above choices of  $l_n$  and  $k_n$  have been made such that  $\pi(z)$  commutes with  $\pi(U_\gamma)$  and  $\pi(V_\gamma)$  and  $\pi(U_\gamma)\pi(V_\gamma) = e^{i2\pi \gamma}\pi(V_\gamma)\pi(U_\gamma)$ . Notice that  $C^*(\mathbb{Z}) \otimes \mathcal{A}_\theta^2$  is a 3-dimensional torus. By its universal property, we can extend  $\pi$  to a  $*$ -representation  $\pi : C^*(\mathbb{Z}) \otimes \mathcal{A}_\theta^2 \rightarrow L_\infty(\mathbb{T}_\theta^2)_U$ . It is obviously trace preserving, hence it extends to complete isometries at all  $L_p$ -levels  $\square$

### 3.3. Applications to multipliers.

**Proposition 3.6.** *Let  $1 \leq p \leq \infty$ , and  $\varphi : \mathbb{Z} \rightarrow \mathbb{C}$  be periodic and  $\theta \notin \mathbb{Q}$ , then*

$$\|M_{\varphi \otimes 1} : L_p(\mathbb{T}_\theta^2) \rightarrow L_p(\mathbb{T}_\theta^2)\| = \|M_\varphi : L_p(\mathbb{T}) \rightarrow L_p(\mathbb{T})\|_{cb}.$$

*Proof.* The proof uses transference again. Let  $N$  be the period of  $\varphi$ . We use the embedding  $\pi$  of  $L_p(\mathbb{T}) \otimes L_p(\mathbb{T}_\theta^2)$  from Proposition 3.5 with  $\gamma = \theta$ . As  $\varphi$  is  $N$ -periodic, the multiplier  $M_{\varphi \otimes 1}$  is well defined and uniformly bounded on  $L_\infty$  and  $L_2$ , we can consider its ultrapower  $(M_{\varphi \otimes 1})_U$  by Lemma 3.4. Take  $a, b, c \in \mathbb{Z}$ , we have by  $N$ -periodicity

$$(M_{\varphi \otimes 1})_U \pi(z^a U_\theta^b V_\theta^c) = (M_{\varphi \otimes 1}(U_\theta^{a+bNl_n} V_\theta^{cl_n}))^\bullet = \varphi(a) \pi(z^a U_\theta^b V_\theta^c).$$

By linearity, continuity and density, we get for all  $x \in L_p(\mathbb{T}) \otimes L_p(\mathbb{T}_\theta^2)$

$$\pi((M_\varphi \otimes Id)x) = (M_{\varphi \otimes 1})_U \pi(x).$$

Hence by Lemma 3.4, we get that  $\|M_\varphi \otimes Id\| \leq \|M_{\varphi \otimes 1}\|$ . But  $L_\infty(\mathbb{T}_\theta^2)$  is the hyperfinite factor, so we can conclude  $\|M_\varphi \otimes Id\| = \|M_\varphi\|_{cb}$ . The other inequality is clear by Proposition 2.1 as

$$\|M_\varphi\|_{cb(L_p(\mathbb{T}))} = \|M_{\varphi \otimes 1}\|_{cb(L_p(\mathbb{T}^2))} = \|M_{\varphi \otimes 1}\|_{cb(L_p(\mathbb{T}_\theta^2))}.$$

$\square$

**Theorem 3.7.** *For any  $\theta \notin \mathbb{Q}$  and  $1 < p \neq 2 < \infty$ , there exists  $\varphi : \mathbb{Z}^2 \rightarrow \mathbb{C}$  such that*

$$\|M_\varphi : L_p(\mathbb{T}^2) \rightarrow L_p(\mathbb{T}^2)\| < \infty \quad \text{but} \quad \|M_\varphi : L_p(\mathbb{T}_\theta^2) \rightarrow L_p(\mathbb{T}_\theta^2)\| = \infty.$$

*Proof.* If not, by the closed graph theorem, there is a constant  $c$  such that

$$\|M_\varphi : L_p(\mathbb{T}_\theta^2) \rightarrow L_p(\mathbb{T}_\theta^2)\| \leq c \|M_\varphi : L_p(\mathbb{T}^2) \rightarrow L_p(\mathbb{T}^2)\|.$$

By Pisier's example, there is a multiplier  $\varphi$  on  $L_p(\mathbb{T})$  that is not completely bounded. By Proposition 3.3, we can find periodic multipliers  $\varphi_n$  on  $L_p(\mathbb{T})$  with  $\|M_{\varphi_n}\| \rightarrow \|M_\varphi\|$  and  $\|M_{\varphi_n}\|_{cb} \rightarrow \infty$ .

Next  $\|M_{\varphi_n \otimes 1} : L_p(\mathbb{T}_\theta^2) \rightarrow L_p(\mathbb{T}_\theta^2)\| \rightarrow \infty$  by Proposition 3.6. But  $\|M_{\varphi_n \otimes 1} : L_p(\mathbb{T}^2) \rightarrow L_p(\mathbb{T}^2)\| \rightarrow \|M_\varphi : L_p(\mathbb{T}) \rightarrow L_p(\mathbb{T})\|$  by Fubini's theorem.  $\square$

**Remark 3.8.** *Actually one can directly construct periodic  $\varphi_n$  from Pisier's example.*

**Remark 3.9.** *It is also easy to see that  $1 < p \neq 2 < \infty$  there exists  $\varphi : \mathbb{Z}^2 \rightarrow \mathbb{C}$  such that*

$$\|M_\varphi : L_p(\mathbb{T}_\theta^2) \rightarrow L_p(\mathbb{T}_\theta^2)\| < \infty \quad \text{but} \quad \|M_\varphi : L_p(\mathbb{T}_\theta^2) \rightarrow L_p(\mathbb{T}_\theta^2)\|_{cb} = \infty.$$

*This also follows directly from Pisier's example as there is a conditional expectation from  $L_\infty(\mathbb{T}_\theta^2)$  to  $L_\infty(\mathbb{T})$  the algebra generated by  $U$ .*

**Remark 3.10.** *When  $\theta = \frac{a}{b} \in \mathbb{Q}$ , then  $L_p(\mathbb{T}_\theta^2) \subset L_p(\mathbb{T}^2, S_p^b)$ ; a possible embedding is given by  $\pi(U_\theta) = b^{-1/p} z_1 \sum_{k=1}^b e^{2i\pi \frac{ak}{b}} e_{k,k}$ ,  $\pi(V_\theta) = b^{-1/p} z_2 \sum_{k=1}^b e_{k,k-1(b)}$  where  $e_{k,l}$  is the canonical basis of  $S_p^b$ . Hence multipliers are automatically completely bounded. Nevertheless, the equivalence constant must behave badly, but we have no quantitative estimates.*

**Remark 3.11.** *The algebra  $L_\infty(\mathbb{T}_\theta^2)$  can be seen as a crossed product  $L_\infty(\mathbb{T}) \rtimes_\theta \mathbb{Z}$  where the action comes from the rotation of angle  $\theta$ . Thus, the above proofs produce a  $\mathbb{Z}$ -equivariant map  $T$  on  $L_p(\mathbb{T})$  such that  $T \rtimes Id$  is unbounded on  $L_p$ .*

**Corollary 3.12.** *For any  $\theta, \gamma \in \mathbb{R} \setminus \mathbb{Q}$  such that  $\theta \notin \gamma\mathbb{Q}$  and  $1 < p \neq 2 < \infty$ , there exists  $\varphi : \mathbb{Z}^2 \rightarrow \mathbb{C}$  such that*

$$\|M_\varphi : L_p(\mathbb{T}_\gamma^2) \rightarrow L_p(\mathbb{T}_\gamma^2)\| < \infty \quad \text{but} \quad \|M_\varphi : L_p(\mathbb{T}_\theta^2) \rightarrow L_p(\mathbb{T}_\theta^2)\| = \infty.$$

*Proof.* This is once again a transference from the multiplier constructed in Theorem 3.7.

As above if the result does not hold, there exists  $c$  with

$$(1) \quad \|M_\psi : L_p(\mathbb{T}_\theta^2) \rightarrow L_p(\mathbb{T}_\theta^2)\| \leq c \|M_\psi : L_p(\mathbb{T}_\gamma^2) \rightarrow L_p(\mathbb{T}_\gamma^2)\|.$$

By the assumption, for all  $n > 0$ , we can find some  $k_n$  such that

$$|e^{i2\pi k_n \gamma} - 1| < \frac{1}{n}, \quad |e^{i2\pi k_n \theta} - e^{i2\pi \theta}| < \frac{1}{n}.$$

Indeed the set  $\mathbb{N}(\theta, \gamma)$  is equidistributed in  $\mathbb{R}^2$  modulo 1.

The subalgebra  $\{U_\gamma, V_\gamma^{k_n}\}''$  is  $L_\infty(\mathbb{T}_{k_n \gamma}^2)$ , the associated conditional expectation corresponds to the multiplier associated to the function  $\psi(k, l) = 1_{k_n | l}$ . And similarly with  $\theta$ . Thus applying (1) to multipliers of the form  $\varphi(k, l) = 1_{k_n | l} f(k, l/k_n)$  for  $f : \mathbb{Z}^2 \rightarrow \mathbb{C}$ , we deduce that

$$(2) \quad \|M_f : L_p(\mathbb{T}_{k_n \theta}^2) \rightarrow L_p(\mathbb{T}_{k_n \theta}^2)\| \leq c \|M_f : L_p(\mathbb{T}_{k_n \gamma}^2) \rightarrow L_p(\mathbb{T}_{k_n \gamma}^2)\|.$$

Let  $\mathcal{P}_d^\alpha = \text{span}\{U_\alpha^k V_\alpha^l : |k|, |l| \leq d\}$ . Going to ultraproducts as in Proposition 3.5 we deduce that there are cb isometric embeddings

$$(3) \quad \pi_\gamma : L_p(\mathbb{T}^2) \rightarrow \prod_{\mathfrak{U}} L_p(\mathbb{T}_{k_n \gamma}^2), \quad \pi_\theta : L_p(\mathbb{T}^2) \rightarrow \prod_{\mathfrak{U}} L_p(\mathbb{T}_{k_n \theta}^2)$$

given by the extension of  $\pi_\gamma(U_0) = (U_{k_n \gamma})^\bullet$ ,  $\pi_\gamma(V_0) = (V_{k_n \gamma})^\bullet$  and  $\pi_\theta(U_0) = (U_{k_n \theta})^\bullet$ ,  $\pi_\theta(V_0) = (V_{k_n \theta})^\bullet$ .

In particular this restricts to  $(\mathcal{P}_d^0, \|\cdot\|_p) = \prod_{\mathfrak{U}} (\mathcal{P}_d^{k_n \gamma}, \|\cdot\|_p)$  given by the ultraproduct of the formal identity maps  $i_n : U_0^k V_0^l \mapsto U_{k_n \gamma}^k V_{k_n \gamma}^l$  (recall that the ultraproduct of  $m$ -dimensional spaces is still  $m$ -dimensional). As the spaces share the same finite dimension, by a compactness argument, we get that  $\lim_{\mathfrak{U}} \|i_n : (\mathcal{P}_d^0, \|\cdot\|_p) \rightarrow (\mathcal{P}_d^{k_n \gamma}, \|\cdot\|_p)\| = \lim_{\mathfrak{U}} \|i_n^{-1} : (\mathcal{P}_d^{k_n \gamma}, \|\cdot\|_p) \rightarrow (\mathcal{P}_d^0, \|\cdot\|_p)\| = 1$ .

Let  $\varphi : \mathbb{Z}^2 \rightarrow \mathbb{C}$  and  $F_d^2 = F_d \otimes F_d$  be the symbol of the 2-dimensional Fejér kernel. As we can write  $M_{F_d^2 \varphi} = i_n \circ M_\varphi \circ i_n^{-1} \circ M_{F_d^2}$ , we get  $\lim_{\mathfrak{U}} \|M_{F_d^2 \varphi} : L_p(\mathbb{T}_{k_n \gamma}^2) \rightarrow L_p(\mathbb{T}_{k_n \gamma}^2)\| \leq \|M_\varphi : L_p(\mathbb{T}^2) \rightarrow L_p(\mathbb{T}^2)\|$ .

Using the isometric embedding  $\pi_\theta$  from (3), the estimate (2), we can consider the ultraproduct of  $M_{F_d^2 \varphi}$  to get  $\|M_{F_d^2 \varphi} : L_p(\mathbb{T}_\theta^2) \rightarrow L_p(\mathbb{T}_\theta^2)\| \leq c \|M_\varphi : L_p(\mathbb{T}^2) \rightarrow L_p(\mathbb{T}^2)\|$ . Letting  $d \rightarrow \infty$  contradicts Theorem 3.7.  $\square$

#### 4. BOUNDED MULTIPLIERS ON $L_\infty$

We conclude by giving an analogue of Theorem 3.7 for  $p = \infty$  (or  $p = 1$  by duality), recall that  $\|M_\varphi : L_\infty(\mathbb{T}_\theta^2) \rightarrow L_\infty(\mathbb{T}_\theta^2)\|_{cb} = \|M_\varphi : L_\infty(\mathbb{T}^2) \rightarrow L_\infty(\mathbb{T}^2)\|$  by Proposition 2.1.

**Proposition 4.1.** *For any  $\theta \notin \mathbb{Q}$ , there exists  $\varphi : \mathbb{Z}^2 \rightarrow \mathbb{C}$  such that*

$$\|M_\varphi : L_\infty(\mathbb{T}_\theta^2) \rightarrow L_\infty(\mathbb{T}_\theta^2)\| < \infty \quad \text{but} \quad \|M_\varphi : L_\infty(\mathbb{T}_\theta^2) \rightarrow L_\infty(\mathbb{T}_\theta^2)\|_{cb} = \infty.$$

*Proof.* Assume the conclusion does not hold, then there is a constant  $c$ , with  $\|M_\varphi\|_{cb} \leq c \|M_\varphi\|$ .

One can construct inductively increasing sequences of integers  $k_n, l_n$  such that  $k_n$  is odd and

$$\forall j < n, \quad |e^{i2\pi \theta k_j l_n} + 1| < \frac{1}{2^n}, \quad |e^{i2\pi \theta k_n l_j} - 1| < \frac{1}{2^n}.$$

Indeed to construct  $l_{n+1}$ , one just need to pick it such that  $|e^{i\pi \theta l_{n+1}} + 1| < \frac{1}{2^n k_n}$  as  $\theta \notin \mathbb{Q}$ . Then  $|e^{i2\pi \theta k_j l_n} + 1| < \frac{1}{2^n}$  follows by taking  $k_j$  powers.

Similarly to get  $k_{n+1}$ , it suffices to choose  $k$  big enough and such that  $|e^{i2\pi (2\theta k + \theta)} - 1| < \frac{1}{2^n l_n}$  as  $2\theta \notin \mathbb{Q}$  and to put  $k_{n+1} = 2k + 1$ .

Put  $e_n = U_\theta^{k_n} V_\theta^{l_n}$ . We have  $e_n e_j = e^{i2\pi \theta k_j l_n} U_\theta^{k_n + k_j} V_\theta^{l_n + l_j}$ , hence

$$\|e_n e_j + e_j e_n\| \leq \frac{1}{2^{n-1}}, \quad \|e_n e_j^* + e_j^* e_n\| \leq \frac{1}{2^{n-1}}.$$

Fix some  $N$ . For any  $a_j \in \mathbb{C}$ , and  $x = \sum_{j=1}^N a_j e_{n+j}$ , we have the following inequalities

$$\|x\|^2 \geq \|\tau(x^* x)\| = \sum_{j=1}^N |a_j|^2$$

$$\|x\|^2 \leq \|x^*x + xx^*\| \leq 2\left\|\sum_{j=1}^N |a_j|^2 1\right\| + \frac{N(N-1)}{2^{n-1}} \max_j |a_j|^2.$$

We deduce that for  $n$  big enough on the subspace  $E_N^n = \text{span}(e_{n+j})_{j=1}^N$  the  $L_\infty$ -norm is smaller than 2 times the  $L_2$ -norm. Since the  $L_\infty$ -norm dominates the  $L_2$ -norm,  $E_N^n$  is then 2-complemented in  $L_\infty(\mathbb{T}_\theta^2)$  by the natural orthogonal projection; this is the Fourier multiplier  $M_\psi$  where  $\psi$  is the characteristic function of  $\{(k_{n+j}, l_{n+j})\}$ . More generally for any choice of sign  $\varepsilon = (\varepsilon_i)$ , the function  $\psi_\varepsilon = \sum_{i=1}^N \varepsilon_i 1_{\{(k_{n+j}, l_{n+j})\}}$  defines a Fourier multiplier on  $L_\infty$  of norm at most 4. Hence we can think of  $\{(k_{n+j}, l_{n+j})_{j \geq 0}\}$  as a kind of generalized hilbertian-Sidon set.

We will use an argument from [4] section 7 to get a contradiction. We must have  $\|M_{\psi_\varepsilon}\|_{cb} \leq 4c$ , hence the sequence  $(e_{n+j})_{j=1}^N$  is  $4c$ -completely unconditional in  $L_\infty(\mathbb{T}_\theta^2)$  and  $4c$ -completely complemented. By duality, the same holds in  $L_1(\mathbb{T}_\theta^2)$ .

Let  $f_i = e_i \otimes \delta_i$  where  $\delta_i$  is the basis of the operator space  $R_N \cap C_N$ . Then, using that all  $e_i$  are unitaries, we get that for matrices  $a_i \in M_d$

$$\left\|\sum_i a_i \otimes f_i\right\|_{M_d \otimes L_\infty(\mathbb{T}_\theta^2) \otimes (R_N \cap C_N)} = \left\|\sum_i a_i \otimes \delta_i\right\|_{M_d \otimes (R_N \cap C_N)} = \max\left\{\left\|\sum_i a_i^* a_i\right\|^{1/2}, \left\|\sum_i a_i a_i^*\right\|^{1/2}\right\}.$$

Moreover by complete unconditionality,  $F_N = \text{span } f_i \subset L_\infty(\mathbb{T}_\theta^2) \otimes (R_N \cap C_N)$  is completely complemented in  $L_\infty(\mathbb{T}_\theta^2) \otimes (R_N \cap C_N)$  with constant  $4c$ . Indeed the projection is just the average over all possible signs  $\mathcal{E}_\varepsilon(M_{\psi_\varepsilon} \otimes T_\varepsilon)$  where  $T_\varepsilon(\delta_i) = \varepsilon_i \delta_i$ . By duality, we deduce that  $F_N^* \approx \text{span } f_i \subset L_1(\mathbb{T}_\theta^2) \hat{\otimes} (R_N + C_N)$ . Next by the noncommutative Khintchine inequalities ([5] Section 8.4) and complete unconditionality, for any matrices  $a_i \in S_1^d$

$$\left\|\sum_i a_i \otimes f_i\right\|_{S_1^d \hat{\otimes} L_1(\mathbb{T}_\theta^2) \hat{\otimes} (R_N + C_N)} \approx \mathcal{E}_\varepsilon \left\|\sum_i a_i \otimes \varepsilon_i e_i\right\|_{S_1^d \hat{\otimes} L_1(\mathbb{T}_\theta^2)} \approx \left\|\sum_i a_i \otimes e_i\right\|_{S_1^d \hat{\otimes} L_1(\mathbb{T}_\theta^2)}.$$

Hence  $(e_i)$  generates  $R_N + C_N$  as an operator space in  $L_1$ . Using duality once again, as  $M_\psi$  is a projection, we get that for some constant  $C$  independent of  $N$ ,  $E_N^n$  is  $C$ -completely isomorphic to  $R_N \cap C_N$ . This contradicts the fact that  $R_N \cap C_N$  has an injectivity constant of order  $\sqrt{N}$  (see [6]).  $\square$

The above argument actually shows that a generalized completely hilbertian-Sidon set must span  $R \cap C$  as an operator space.

There is a possible variant of the proof. The set  $\{e_n, n \geq 1\}$  actually generates a copy of  $\ell_2$  in  $L_\infty(\mathbb{T}_\theta^2)$  and is complemented. By choosing in the construction very rapidly increasing sequences  $(k_n, l_n)_{n \geq 1}$ , one could also make  $\{(k_{n+j}, l_{n+j})_{j \geq 0}\}$  a Sidon set in  $L_\infty(\mathbb{T}^2)$ . But infinite Sidon sets are not complemented, this also leads to a contradiction.

Using the embedding and techniques of Corollary 3.12 with the injectivity of  $L_\infty(\mathbb{T}^2)$ , one can also prove that for any  $\theta, \gamma \in \mathbb{R} \setminus \mathbb{Q}$  such that  $\theta \notin \gamma\mathbb{Q}$ , there is a multiplier  $\varphi$  such that

$$\|M_\varphi : L_\infty(\mathbb{T}_\theta^2) \rightarrow L_\infty(\mathbb{T}_\theta^2)\| < \infty \quad \text{but} \quad \|M_\varphi : L_\infty(\mathbb{T}_\gamma^2) \rightarrow L_\infty(\mathbb{T}_\gamma^2)\| = \infty.$$

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